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COMMENTS ON THE PAPER: THEORETICAL FOUNDATIONS OF THE FINITE ELEMENT METHOD†

IN THE summer of 1965 following some clues in the literature [1] and also some early indications in the experiments on the Zienkiewicz plate bending triangle [2], certain definite conclusions were reached concerning the convergence of nonconforming plate elements. The numerical experiments and the theory that subsequently reinforced the conclusions are described in the addendum to Ref. [2]. The experiments are now described less succinctly, as their import appears to have been generally misunderstood.

If we postulate that a particular solution converges as the mesh is refined, giving smoothly varying stresses, we may presume also that the finite element representation gives sensibly constant bending stress within each element, and hence sensibly constant curvature. This is why, in engineering terms, displacement functions must be "complete"[3]. However, early experiments suggested that the responsibilities were not entirely fulfilled unless conformity was also imposed. It was fortunate that four elements were simultaneously under test [2], one of which, the Zienkiewicz version, was nonconforming, the other three conforming. All four gave the states of constant curvature under suitable nodal action, and all four were derived from a single shape function routine with options, by numerical integration. In two cases this was exact. The routine was checked thoroughly by various numerical tests.

The first indication of the strange behaviour of the nonconforming triangle was when mesh (a) of Fig. 1 responded correctly to distributed bending moment applied at the free

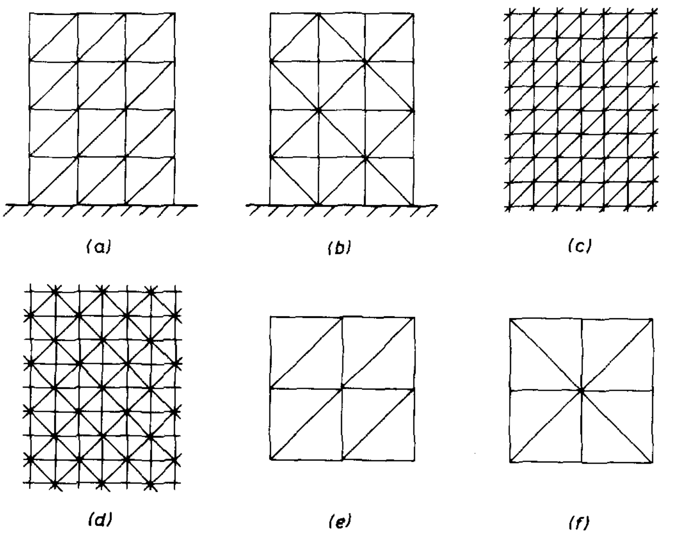

FIG. 1. The mesh pattern on which experiments demonstrated convergence or non-convergence.

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end whereas (b) did not. The evidence was already irrefutable, for the conforming triangles all behaved correctly in both mesh (a) and (b). This evidence suggested that a mesh of type (c), composed of three sets of parallel lines, gives convergence, whereas mesh (d), composed of "union jack" figures, would not give convergence, however fine the mesh. This suggestion may be easily and decisively checked, by imposing a state of constant curvature on the basic figures which, replicated in both directions, give (c) or (d). Thus, when the external nodes of (e) and (f) are given slopes and deflections corresponding to states of constant curvature, either the elements behave, or they misbehave. That is, the internal node either takes up its correct slopes and deflection corresponding to the required state of constant curvature, or it does not. With the non-conforming triangle, (e) behaved but (f) did not. (Despite this result, the element is useful and gives accurate results in practice.)

These observations did not complete the research. The theory they suggested showed that, provided the element has certain symmetries possessed by every practical element, mesh (c) converges, regardless of the angles of the three sets of parallel lines. This property extends also to the general anisotropic case, and it even extends to the case where the integration formula is not accurate enough to give correct stiffness coefficients [4]. As a bonus, the theory predicts correctly [1] that a mesh of equal Melosh rectangles converges.

These observations and theoretical predictions seem to be at variance with the conclusions of Ref. [2]. It is the purpose of the present note to clarify what has been proved.

The important point is that the convergence proof presented in Ref. [2] depends on u_{cn} belonging to C_{1n}, as indicated in the text.

The analogy with the case of Poisson's equation permits the conclusion that u_{cn} belongs to C_{1n} in the case when operator A is a second order differential operator and the components of f_n (body force density components in the elastic case) corresponding to the successive approximate solutions, remain continuous and bounded within each element as the size decreases indefinitely. This is the case in two- and three-dimensional elasticity, and hence also for plates, shells and beams if the transverse shear deformation has not been neglected.

The situation changes however if the order of the highest derivative of u_i contained in A exceeds p_i+1, because the boundedness of the components of f_n does not then ensure the boundedness of the (p_i+1)th derivatives of the field component u_i. The (p_i+1)th derivatives can, for instance, be constant within each element but unbounded as the size decreases indefinitely, whereas the (p_i+2)th derivatives vanish and are thus bounded.

Such is the case in the elastic theories which result from neglecting the transverse shear deformation. The appropriate convergence conditions can be established by resorting to the convergence conditions of the corresponding theories in which the transverse shear deformation is not neglected.

In the simplified theory of thin plates, for instance, A is a fourth order operator and (p_i+1) is equal to 3. The convergence conditions are, completeness, and the condition that the third derivatives of the transverse displacement, corresponding to the successive approximate solutions, remain bounded within each element as the size decreases (see Ref. [5]).

The union jack mesh (d) does not give convergence because the third derivatives of the transverse displacement are unbounded as the size of the element decreases. Such derivatives remain bounded in mesh (c): convergence is thus obtained. If conformity were guaranteed, completeness would be sufficient for convergence regardless of the mesh pattern. Completeness and the boundedness of the (p_i+1)th derivatives corresponding to the successive approximate solutions appear to be sufficient convergence conditions in the general case.

Such conditions can at least be shown to be sufficient if a type of element exists with the same general shape and nodes as the one which is to be investigated, but meeting both completeness and conformity conditions. Let u_{an}^* be the field generated by such an element, when the nodes and nodal displacements are the same as those corresponding to u_{an}. The same argument that was used to prove the completeness criterion in section 9 of Ref. [3] may indeed be used again (if the (p_i+1)th derivatives of u_{an} are bounded) to show that the distance between the two fields u_{an}^* and u_{an} can be smaller than a given positive and arbitrarily small number. As u_{an} is a compatible field, the inequality (116) of Ref. [3] remains valid and the convergence proof continues as in Ref. [3].

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